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A TRAFFIC COUNTING DISTRIBUTION

by

Robert M. Oliver

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A TRAFFIC COUNTING DISTRIBUTION

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ABSTRACT

A traffic counting distribution is derived in which a minimum spacing or headway between units of traffic is taken into account. A comparison is made between this probability distribution and the Type I Counter distribution derived by W. Feller (3). Explicit expressions are derived for the mean and variance of count as well as the probability that the interval of interest is completely filled by vehicles.

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Research Report 7

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A TRAFFIC COUNTING DISTRIBUTION

0. Introduction

Several authors⁽¹⁾⁽⁵⁾⁽⁷⁾ have suggested the use of traffic counting distributions in which a minimum spacing or headway between units of traffic is taken into account. Observers of medium and high-density traffic flows along highways, freeways and air lanes have evidence to support such a claim. In the case of road traffic a minimum spacing may be governed by the finite car sizes, the minimum headways by the velocity and spacing characteristics of drivers. For reasons of safety airplanes may have to be separated by a minimum space or time interval.

While at least one author⁽⁵⁾ has pointed out that the relative merits of minimum spacings are evident in the measurement of inter-vehicle distances, this may not be the case in measurements of headways, i. e., the time between the passing of two vehicles. Leaving aside, momentarily, the question of the domain of definition of the velocity distribution and its fundamental effect on space-time measurements, the author of this paper feels that the probability distribution of inter-vehicle headways which includes a minimum headway may serve as a valuable approximation to those cases where small headways are merely improbable.

A number of probability distributions of inter-vehicle spacings or headways have been proposed; in many cases the corresponding counting distributions have been found. When the inter-vehicle spacing is exponentially distributed, the discrete counting distribution is the well known Poisson case. When the probability density distribution of inter-vehicle spacings is Erlang, Gamma or Type III, the corresponding counting distribution is that of the Generalized Poisson⁽⁸⁾ (one starts to count with the passing of a vehicle) or the state probabilities discussed in a paper by Jewell⁽⁶⁾ (one starts to count at a random instant of time). In this paper the author obtains solutions for the discrete probability distributions and the average and variance of vehicle counts in an interval when the inter-vehicle spacings have the probability density distribution of Equation (1).

1. The State Probabilities

Let the probability distribution of the spacing between units of traffic be given by

$$a(x) = 0 \quad 0 \leq x \leq x_0 \quad (1a)$$

$$= \mu e^{-\mu(x-x_0)} \quad x_0 \leq x \quad (1b)$$

i.e., the probability than an event occurs in dx is μdx provided x is greater than a minimum spacing x_0 . By standard arguments⁽⁴⁾ which neglect terms proportional to square or higher powers of dx we find that the Chapman-Kolmogorov equations for the state probability $p_n(x)$ of a count of n events in an interval x is given by*

$$\frac{dp_0}{dx} = 1 \quad 0 \leq x \leq x_0 \quad (2a)$$

$$= -\mu p_0 \quad x_0 \leq x \quad (2b)$$

$$\frac{dp_n}{dx} = 0 \quad 0 \leq x \leq nx_0 \quad (3a)$$

$$= \mu p_{n-1} \quad nx_0 \leq x \leq (n+1)x_0 \quad (3b)$$

$$= \mu p_{n-1} - \mu p_n \quad (n+1)x_0 \leq x \quad (3c)$$

In solving these equations of the birth process one can obtain the constants of integration by making use of the condition that the sum of the state probabilities add to one. For a given interval x and minimum spacing, x_0 , the sum extends over a finite number (say, $n = N$) which

* $p_n(x)$ corresponds to $A_n(x)$ in Reference 6; i.e., the counting origin starts with the passing of a vehicle.

represents the fact that a maximum number of events can occur in the interval x . N is just the integral part of xx_0^{-1} which we denote by square brackets,

$$N = \left[xx_0^{-1} \right] \quad (4a)$$

Hence,

$$\sum_{n=0}^N p_n = 1 \quad (4b)$$

Although the algebra becomes involved for $n > 1$, the solutions of Equations (2), (3) are obtained in a straight-forward manner. The solution of $p_0(x)$ is 1 in the interval $(0, x_0)$ and $e^{-\mu(x-x_0)}$ for $x \geq x_0$. This exponential term is just the probability that the spacing between units of traffic is greater than x , i. e., the tail of the distribution of $a(x)$ in Equation (1).

The probability of one event in $(0, x_0)$ is zero since units of traffic must be separated by at least x_0 . The probability of one unit in $x(x_0 \leq x < 2x_0)$ is just $1 - e^{-\mu(x-x_0)}$ since there can only be zero or one unit in an interval less than $2x_0$ in length. The probability of one unit in $x(2x_0 \leq x)$ is the solution of (3b) when one substitutes 1 for n and $e^{-\mu(x-x_0)}$ for $p_0(x)$. In summary we find that

$$p_1(y) = 0 \quad 0 \leq y \leq x_0 \quad (5a)$$

$$p_1(y) = 1 - e^{-\mu(y-x_0)} \quad x_0 \leq y < 2x_0 \quad (5b)$$

$$= \mu(y - 2x_0) e^{-\mu(y-2x_0)} + e^{-\mu(y-2x_0)} - e^{-\mu(y-x_0)} \quad 2x_0 \leq y \quad (5c)$$

In general we find that the probability distribution is composed of three parts:

$$p_n(y) = 0 \quad 0 \leq y < nx_0 \quad (6a)$$

$$= \gamma \left[n; \mu(y - nx_0) \right] / (n-1)! \quad nx_0 \leq y < (n+1)x_0 \quad (6b)$$

$$= \gamma \left[n; \mu(y - nx_0) \right] / (n-1)! \quad (n+1)x_0 \leq y$$

$$- \gamma \left[n+1; \mu(y - (n+1)x_0) \right] / n! \quad (6c)$$

where $\gamma(n; x)$ is the incomplete Gamma function of order n and argument x ⁽²⁾. The discrete distribution $p_n(y)$ is continuous over the region $0 \leq y < \infty$ and is plotted in Figure 1 for several values of $N = [yx_0^{-1}]$ and average count $\bar{n}(y)$.

W. Feller⁽³⁾ has pointed out that the probability distribution $p_n(y)$ can be expressed as the difference between two expressions $F_{n-1}(y)$ and $F_n(y)$ which are, respectively, the probability that the interval between the n^{th} and $(n+1)^{\text{st}}$ event is less than or equal to y . Since we are considering counting intervals which begin with the passing of a traffic unit, Feller's expression for the probability that the time to the next event is less than or equal to x becomes*

$$F_0(x) = 0 \quad 0 \leq x \leq x_0 \quad (7a)$$

$$= 1 - e^{-\mu(x-x_0)} \quad x \geq x_0 \quad (7b)$$

and the probability that the time between the passing of the n^{th} and $(n+1)^{\text{st}}$ unit is less than or equal to y is just

* In this expression and the ones which immediately follow, Feller uses $F_0(x) = 1 - e^{-\mu x}$ $x \geq 0$, i. e., a counting origin randomly placed outside the minimum separation interval x_0 .

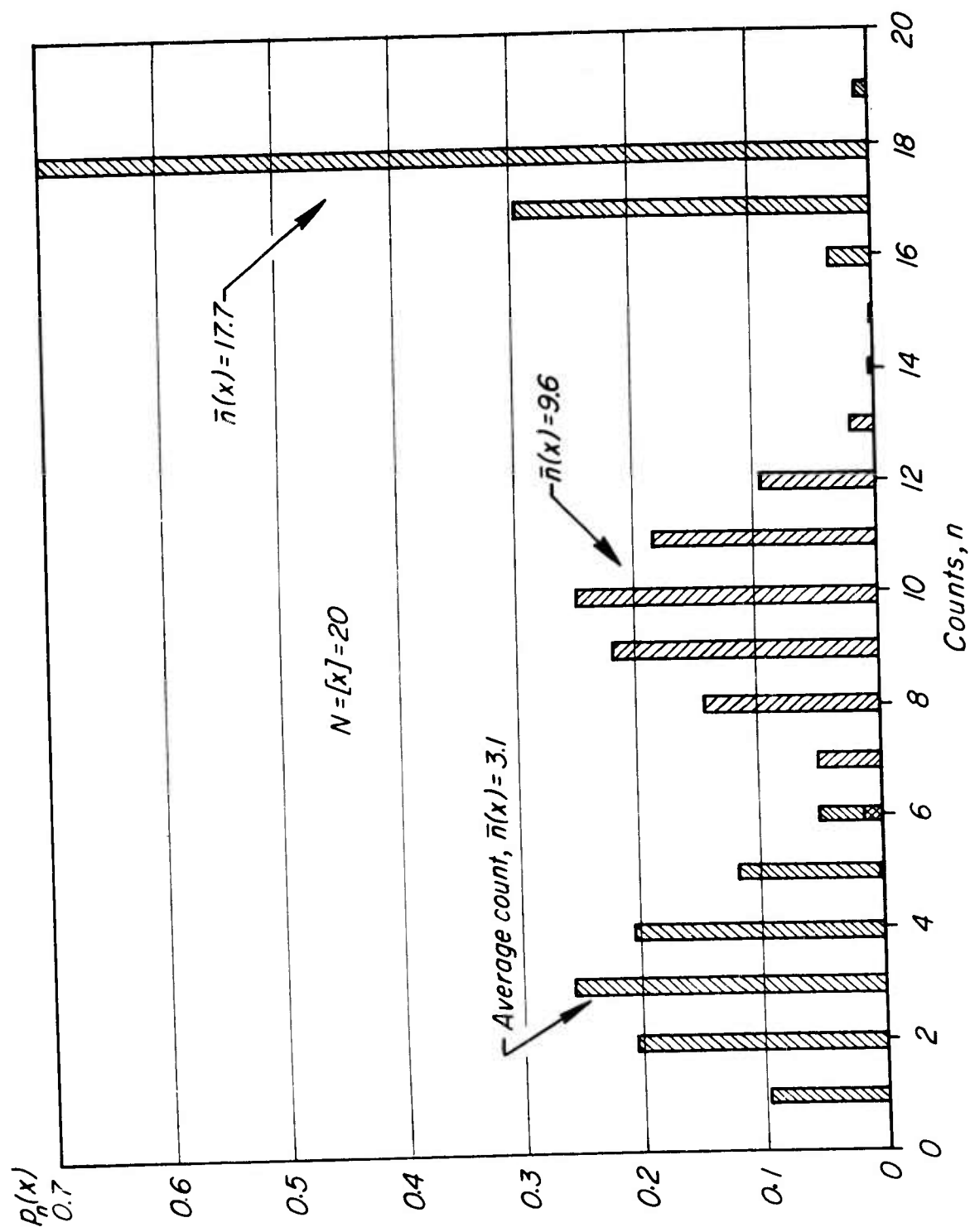


Fig. 1 - THE PROBABILITY OF n COUNTS IN \bar{x} AS A FUNCTION OF n .

$$F_n(y) = \int_0^y F_{n-1}(y-x) dF_0(x) \quad (8a)$$

$$= \int_{x_0}^y \mu F_{n-1}(y-x) e^{-\mu x} dx \quad (8b)$$

$$= \gamma[n+1; \mu(y-(n+1)x_0)] / n! \quad (8c)$$

The difference, $F_{n-1}(y) - F_n(y)$ is just Equation (6b).*

2. Average and Variance of Traffic Counts

Standard renewal arguments can be invoked to derive the average and variance of the distribution of Equation (6). They can also be derived by inversion of the derivatives of the double transform

$$P(s; z) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-sy} z^n p_n(y) dy \quad (9)$$

Fortunately,⁽³⁾⁽⁶⁾ the expression for $P(s; z)$ and moments can be obtained in terms of the Laplace transform $\tilde{a}(s)$ of the inter-event distribution, $a(x)$, in Equation (1):

$$\tilde{a}(s) = \int_0^{\infty} e^{-sx} a(x) dx = \frac{\mu e^{-sx_0}}{\mu + s} \quad (10)$$

The expression for $P(s; z)$ is then **

* In the notation of incomplete Poisson sums, $\frac{\gamma(n; x)}{(n-1)!} = \sum_{j=n}^{\infty} \frac{e^{-x} x^j}{j!} = 1 - E_{n-1}(x)$,

the expressions of Equations (6) and (8c) are of the form $1 - E_n(\mu y - (n+1)x_0)$.

** On expanding $P(s; z)$ in powers of z we obtain the Laplace transform of $p_n(y)$, with inverse transform again equal to the solutions of Equation (6).

$$\frac{1}{s} \left(\frac{s + \mu - \mu e^{-sx_0}}{s + \mu - \mu z e^{-sx_0}} \right) \quad (11)$$

and the Laplace transform of the mean count, $\tilde{n}(s)$ is just

$$\tilde{n}(s) = \tilde{a}(s) [s - s\tilde{a}(s)]^{-1} \quad (12a)$$

$$= \mu e^{-sx_0} [s^2 + \mu s - \mu s e^{-sx_0}]^{-1} \quad (12b)$$

with inverse transform equal to the average count in y

$$\bar{n}(y) = \sum_{j=1}^N \frac{\gamma[j; \mu(y - jx_0)]}{(j-1)!} \quad (13a)$$

By similar arguments the variance is found to be

$$\text{Var}(n(y)) = \sum_{j=1}^N \frac{2j-1}{(j-1)!} \gamma[j; \mu(y - jx_0)] \quad (13b)$$

Figure 2 is a plot of the variance versus the mean for values of $N = 5, 10, 20$. The maximum average counts are 4, 9, 19. This maximum, $N-1$, is due to our choice of origins where one unit of traffic passes just before time zero. The distribution, $p_n(x)$, describes the probability of n events in addition to the one counted at time zero; hence, the maximum value of $\bar{n}(x)$ reflects this feature of being one less than the integral part of xx_0^{-1} .

Several interesting properties of the variance to mean ratio can be observed. In the first place we note that for low average counts in an interval x the limiting variance to mean ratio is unity, i.e., that of the Poisson distribution. When average spacings between cars are much larger

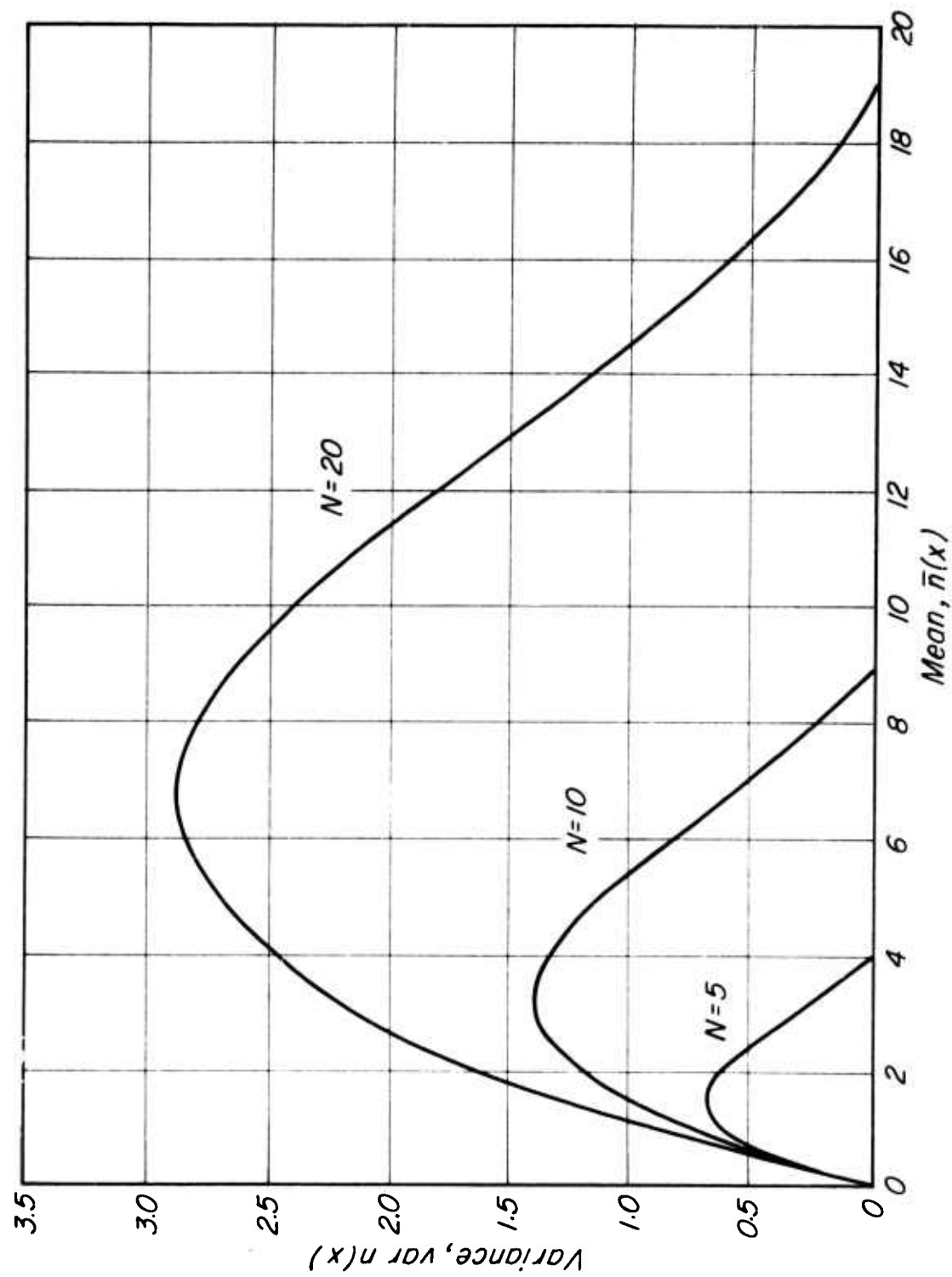


Fig.2 - THE VARIANCE VERSUS THE MEAN.

than the minimum spacing x_0 the low-density Poisson counts are obtained. As the mean count increases the variance also increases, reaches a maximum and then decreases to zero. $\text{Var } n(x)$ is always less than $\bar{n}(x)$ and the probability distribution $p_n(x)$ has a "maximum-packing" property in the sense that the variance of traffic counts is zero when the mean count in x equals the maximum count $N = [\bar{n}x_0^{-1}]$. As we have already seen in Equation (6) $p_n(x) = 0$ for $n > N$ and in the limit of large $\bar{n}(x)$, $p_N(x) = 1$.

3. The Probability of "Maximum-Pack"

It is clear from the physical arguments and from Equation (6) that there is a non-zero probability $p_n(y)$ of counting the maximum number, N , of vehicles which can be packed into an interval $y(Nx_0 \leq y \leq (N+1)x_0)$. The probability $p_n(Nx_0)$ is identically zero since counting begins with the passing of a vehicle. However, if we look at an interval of length Nx_0 with randomly chosen origin, the probability $q_N(Nx_0)^*$ of counting the maximum number in Nx_0 is the convolution of the probability that the first one appears at s and the probability $p_{N-1}(Nx_0 - s)$ that $(N-1)$ vehicles appear in the remainder of the interval,

$$q_N(Nx_0) = (x_0 + \mu^{-1})^{-1} \int_0^{Nx_0} p_{N-1}(Nx_0 - s) \int_s^\infty a(x) dx ds \quad (14a)$$

$$q_N(Nx_0) = (x_0 + \mu^{-1})^{-1} \int_0^{x_0} \frac{\gamma[N-1; \mu(x_0 - s)]}{(N-1)!} ds \quad (14b)$$

* $q_n(x)$ corresponds to $U_n(x)$ in Reference 6; i. e., a randomly chosen counting origin.

$$= (1 + \mu x_0)^{-1} \sum_{j=N-1}^{\infty} \sum_{i=j+1}^{\infty} \frac{e^{-\mu x_0} (\mu x_0)^i}{i!} \quad (14c)$$

In two limiting cases we note that

$$\lim_{\mu \rightarrow \infty} q_N(Nx_0) = 1 \quad (15a)$$

and

$$q_1(x_0) = \frac{\mu x_0}{1 + \mu x_0} \quad (15b)$$

The first of these two equations restates the rather obvious physical fact that in the case of regularly spaced traffic the probability of maximum pack equals unity. The second expression can also be derived by noting that the probability of finding one vehicle in x_0 is equal to the fraction of the average inter-vehicle spacing occupied by an interval of length x_0 or $x_0(x_0 + \mu^{-1})^{-1} = \mu x_0(1 + \mu x_0)^{-1}$. Figure 3 is a plot of the probability of maximum pack, $q_N(Nx_0)$, as a function of μx_0 , the ratio of the minimum headway to the average, $1/\mu$, of the exponential portion of the inter-event distribution of Equation (1).

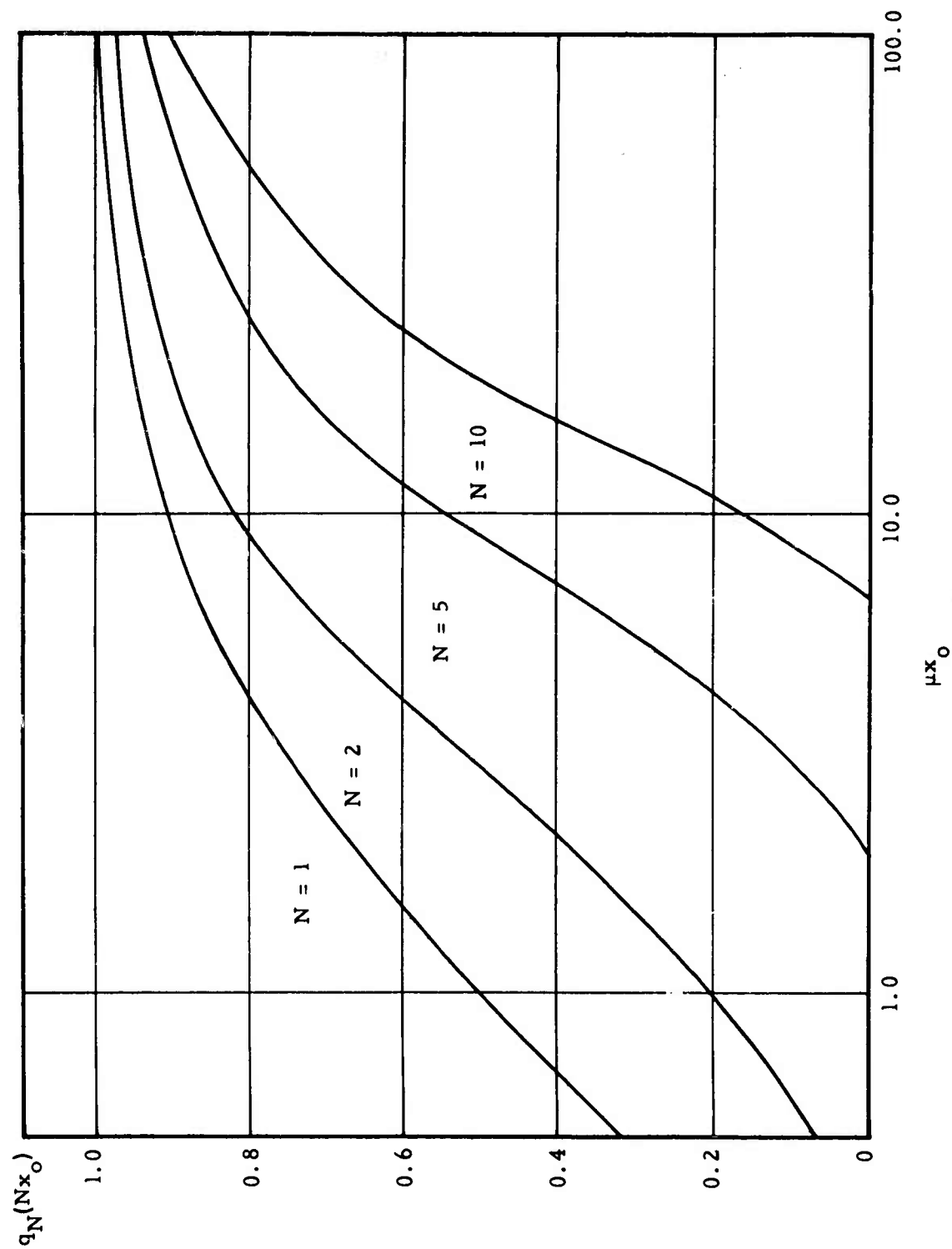


Fig. 3 -- THE PROBABILITY OF "MAXIMUM-PACK."

SUMMARY

Figure 4 is a plot of two experimental curves of the cumulative, $A_0(t)$, of the probability distribution, $a(t)$, of Equation (1).

Figure 4(a) is a plot of the fraction of cars east-bound on Upper Market Street having headways greater than t seconds. These experimental data were obtained from the living room of 251 Upper Terrace, San Francisco 17, in September 1960.

Figure 4(b) is a plot of the cumulative of the distribution of inter-plane arrival times over Runway 13L of the Midway, Chicago Airport. VFR conditions were observed on the 6th of February 1959 when these data were taken.

With the kind permission of P. K. Kinzbruner I am reprinting one of his experimental curves measuring the fraction of cars having inter-vehicle headways greater than a given size. These data were obtained in March and April of 1960 in Boston, Massachusetts. Further details of the experiments are available in his thesis.⁽¹⁾

The fact that the expressions for the counting distributions also support other experiments not reported here suggests that the effect of minimum inter-vehicle spacings and headways may be important in medium and high-density traffic flows.

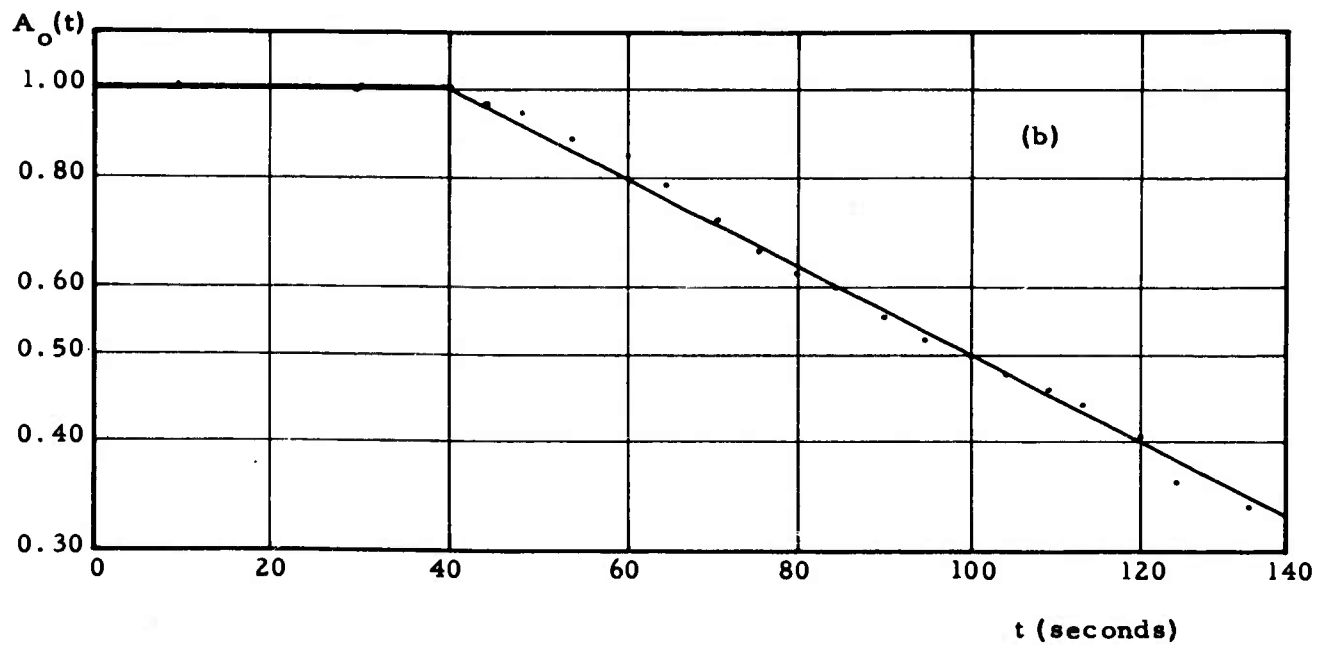
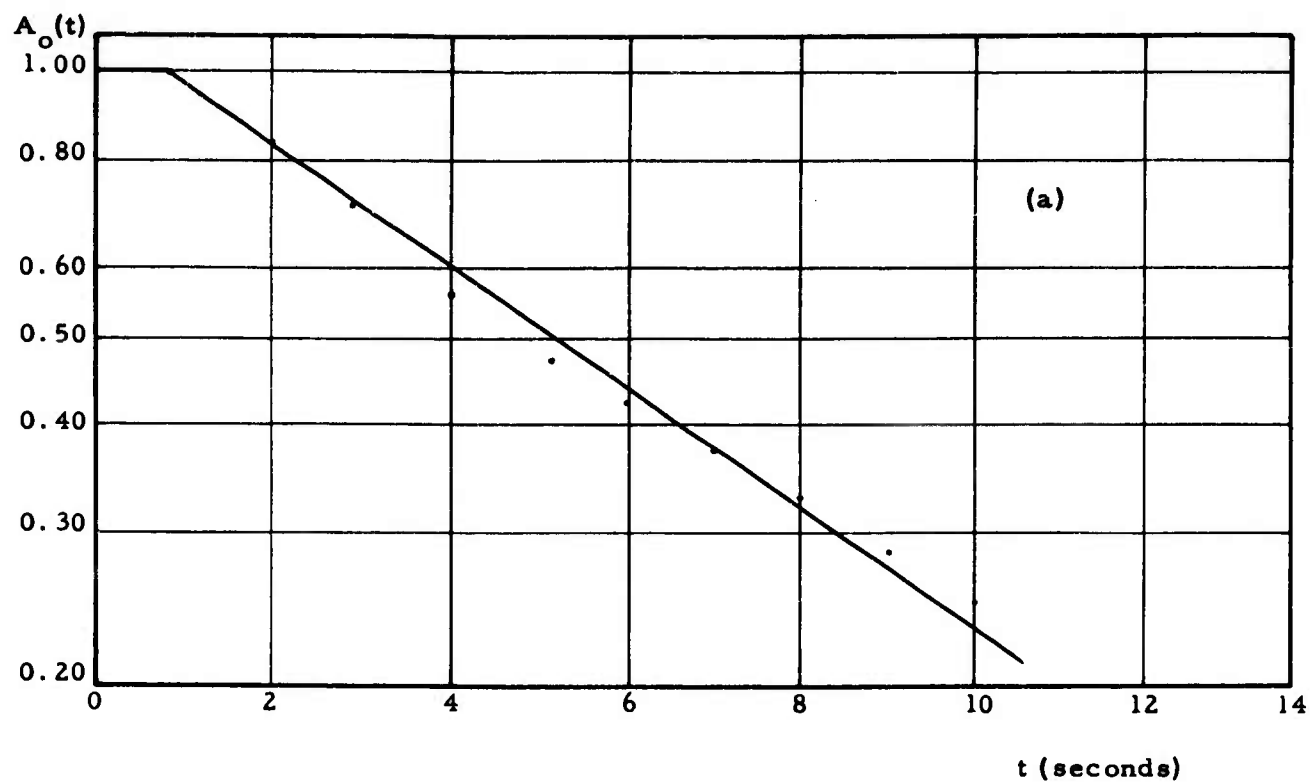


Fig. 4 -- PROBABILITY, $A_o(t)$, HEADWAYS BETWEEN CARS
(a) AND PLANES (b) GREATER THAN t SECONDS. (Semi-log plot.)

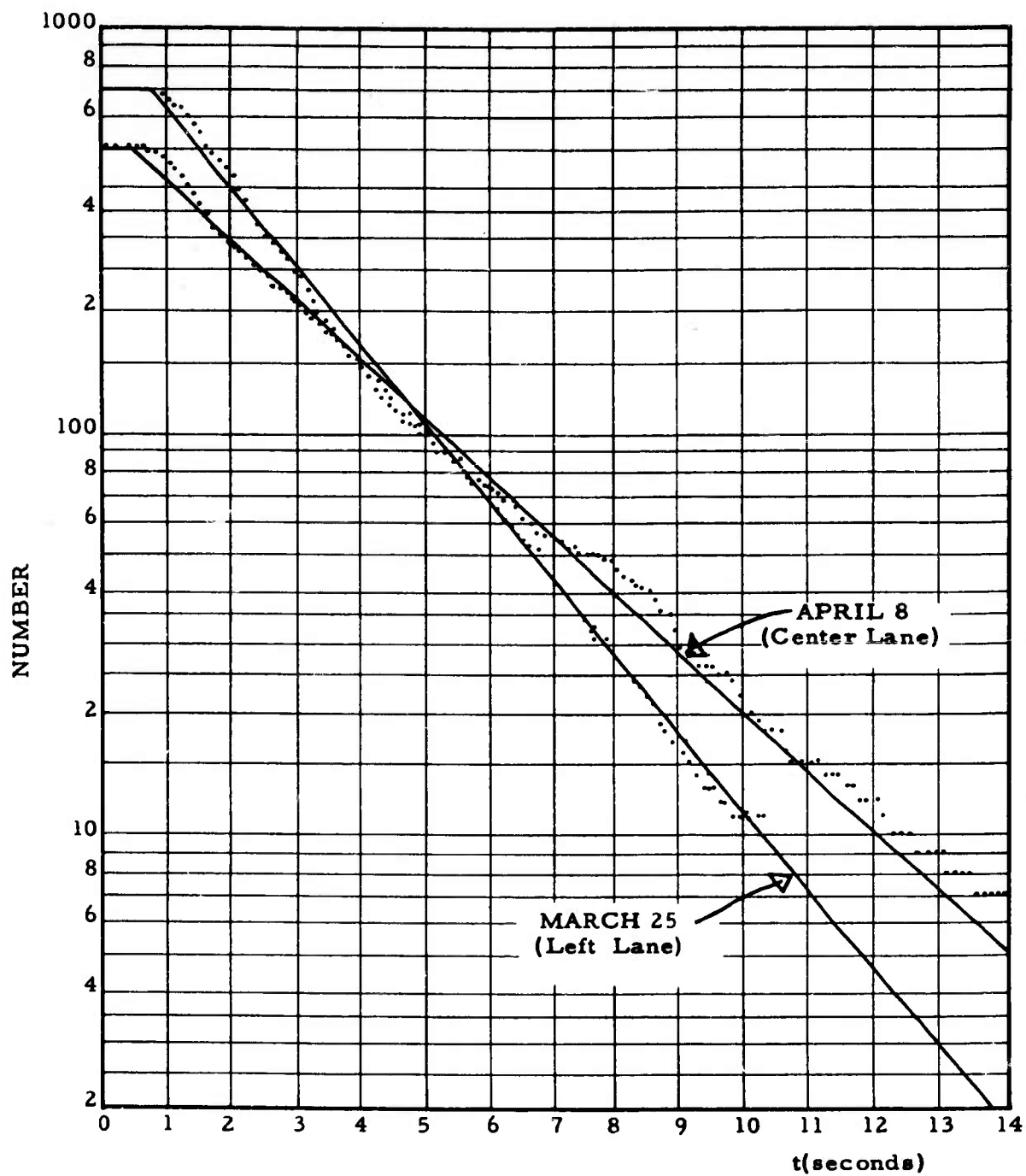


Fig. 5 -- NUMBER OF INTER-ARRIVAL TIMES GREATER THAN t SECONDS.

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